Symbolic analysis of generalized synchronization of chaos

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An approach that uses symbolic analysis is presented for testing generalized synchronization. Generalized synchronization appears when conditional entropy has a sharp minimum. In order to demonstrate how this method works we applied it to the cases of Rössler and Lorenz systems. Our results appear to be robust when external noise is added. [S1063-651X(97)10811-X]

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Cooperation behavior of chaotic dynamical systems and, in particular, synchronization phenomena have received much attention recently. It would seem to play an important role in the ability of complex nonlinear oscillators, such as neurons, to act cooperatively in the performance of various functions. Nevertheless, the notion of synchronization itself lacks a unique interpretation [4]. Generally, synchronization of chaos is understood as a regime in which two coupled chaotic systems exhibit identical, but still chaotic, oscillations [1]. Recently, the concept of synchronization has been extended to two cases. One is called generalized synchronization of chaos [2,3] where the driving and response systems are different. It equates dynamical variables from one subsystem with a function of the variables of another subsystem and exists in directionally coupled chaotic systems. The other is called phase synchronization [4–6] where the systems flow synchronizing in the phase but with different sizes and/or positions. There is a both-way coupling in phase synchronization. In this paper, we call these two cases generalized synchronization. Using the method of symbolic analysis, we show that the conditional entropy has a sharp minimum with the shift of time parameter n_0 when the generalized synchronization is implemented. It means that we do not need to determine a complicated generalized synchronization relationship and the appearance of the sharp minimum can be used as a criterion of generalized synchronization. The advantages of our method are that it does not require an auxiliary system [3] and it works well in all kinds of synchronization.

Consider two systems

$$\dot{x}=f(x,\alpha y), \quad \dot{y}=g(y,\beta x), \tag{1}$$

where x and y denote two systems, respectively, and α and β represent the coupling interaction. If there are some relations between functionals of two processes due to interaction, we say that there is a generalized synchronization between the two systems of Eq. (1). Take two time series from Eq. (1),

$$x_i(n)=x_i(t_0+n\tau), \quad y_j(n)=y_j(t_0+n\tau). \tag{2}$$

Here x_i denotes the ith variable of vector x and y_j the jth variable of vector y, $n=0,1,2,\ldots$. In the computations below we choose $\tau=1$. In order to find the relation between time series $x_i(n)$ and $y_j(n)$ in complex dynamical process, we need to use some coarse representation. That is, one has to substitute actual signals $x_i(n),y_j(n)$ with their symbolic representation. Recently, Ref. [7] gave a symbolic analysis of different chaotic signals in a same system, for example, the time records of $x(t)$ and $z(t)$ generated by the Lorenz model. However, the approach of locating the critical points in Ref. [7] is very complex. In this paper, we will give a simple approach to determine the critical point. And our results show that it is useful in explaining generalized synchronization.

In studing the symbolic dynamics of systems described by differential equations, we have given a useful method [8] to determine the Poincaré section. Its idea is as follows. Testing the phase portrait of the chaotic attractor, one can find a plane where all the interesting trajectories intersect it. This

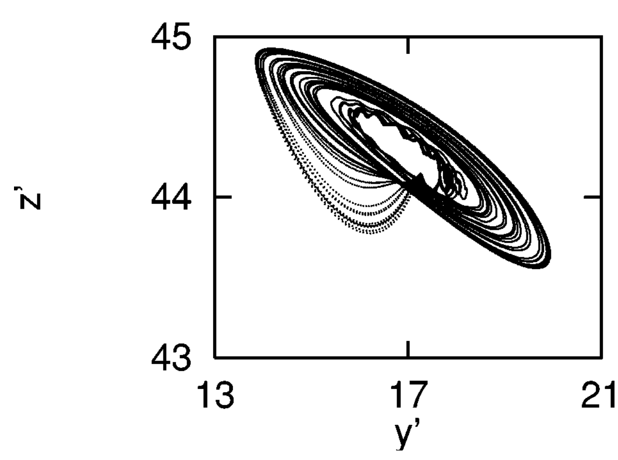

FIG. 1. Phase portrait of Lorenz system when the coupling strength $g=10$.

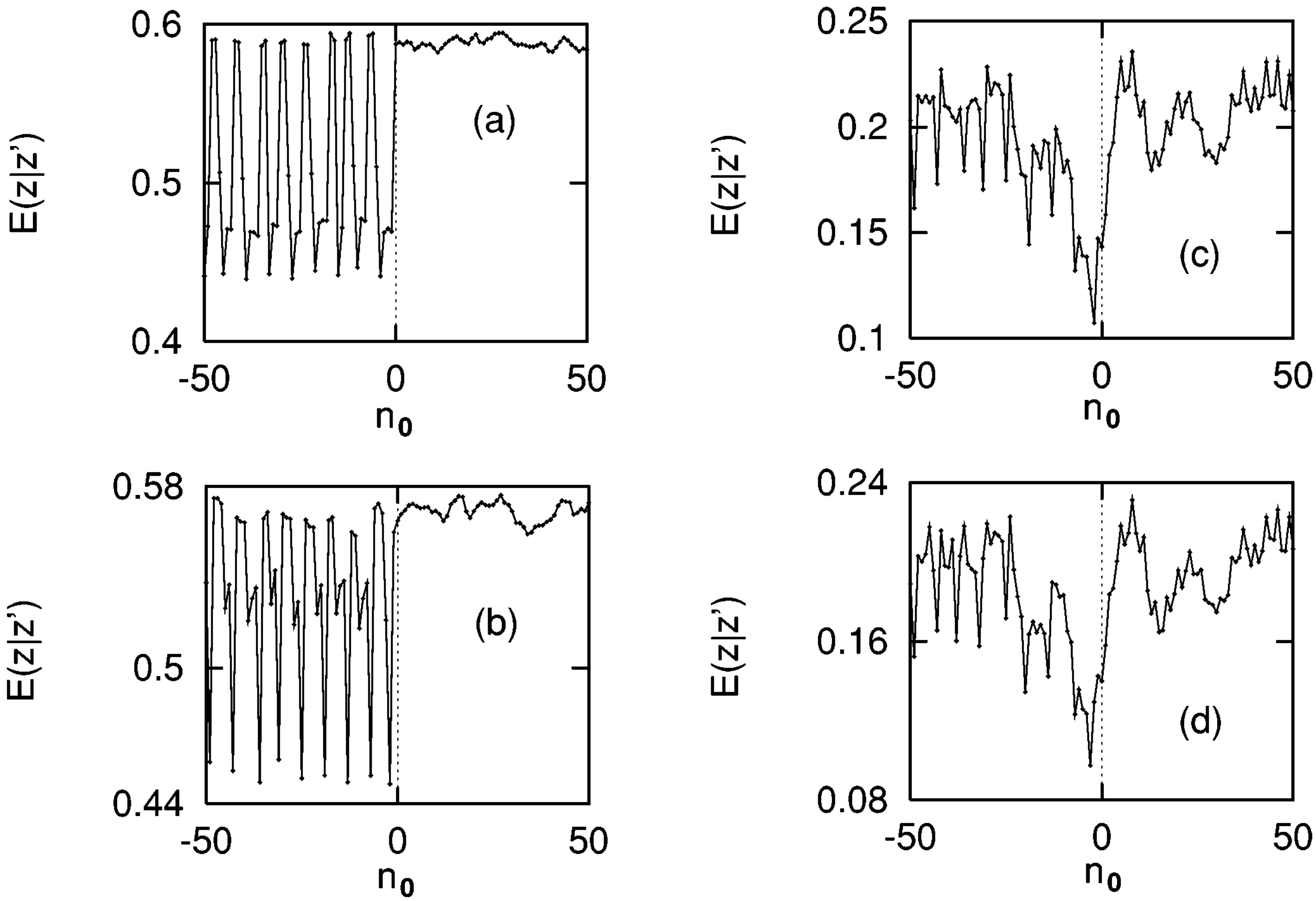

FIG. 2. The conditional entropy $E(z|z')$ where $z_c=0.05$. (a) $g=0$, $z_{c'}=45.0$; (b) $g=3$, $z_{c'}=45.0$; (c) $g=4.9$, $z_{c'}=44.6$; (d) $g=10$, $z_{c'}=44.3$.

plane usually contains an unstable fixed point. Taking this plane as Poincaré section, the points on the plane will demonstrate some behavior of the cubic map. Hence the symbolic dynamics can be set up according to the map. Here we use this plane as a partition plane and give some symbol S_i to represent the time series of Eq. (2). We find that this partition plane will let the entropy of the system approach maximum. The value of S_i is one when the time series of Eq. (2) is above the partition plane and zero when below the partition plane. The resulting long symbolic series we partition into short sequences of a given length L ($L=5$ in the example below), and identify every short sequence uniquely by just one integer [9,10]

$$l=\sum_{i=1}^{L} 2^{L-i}S_i . \tag{3}$$

The sequences l can be used for symbolic coarse graining of the phase space of the dynamical system. Now we represent the time series $x_i(n)$ and $y_j(n)$ as symbolic states $l_x(n)$ and $l_y(n)$. If there is no relation between $x_i(n)$ and $y_j(n)$, the evolution of $l_x(n)$ and $l_y(n)$ states is not correlated. On the other hand, if there is generalized synchronization between $x_i(n)$ and $y_j(n)$, we will observe some relationship between $l_x(n)$ and $l_y(n)$. We can easily destroy such a correlation by time shifting: $l_x(n),l_y(n+n_0)$. To confirm this effect we compute the conditional entropy [7]

$$E(y|x)=-\frac{1}{N_l}\sum_{l_x}\frac{1}{L}\sum_{l_y|l_x} P(l_y(n+n_0)|l_x(n)) \times \ln P(l_y(n+n_0)|l_x(n)), \tag{4}$$

where $P(l_y|l_x)$ is a conditional probability for the variable y_j to occupy state l_y while the variable x_i occupies state l_x, N_l is the total number of different l_x sequences; the first summation in Eq. (4) is done over all dynamically accessible l_y states and fixed l_x states.

As an application, we first consider two different systems. The drive system is the Rössler model,

$$\dot{x}(t)=-[y(t)+z(t)], \quad \dot{y}(t)=x(t)+0.2y(t),$$

$$\dot{z}(t)=0.2+z(t)[x(t)-u], \tag{5}$$

and the response system is the Lorenz model,

$$\dot{x}'(t)=\sigma[y'(t)-x'(t)]-g[x'(t)-x(t)],$$

$$\dot{y}'(t)=-x'(t)z'(t)+rx'(t)-y'(t),$$

$$\dot{z}'(t)=x'(t)y'(t)-bz'(t), \tag{6}$$

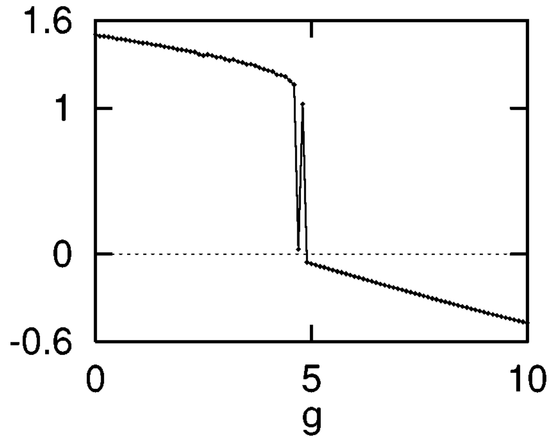

FIG. 3. The largest conditional Lyapunov exponent of the response system versus the coupling strength g.

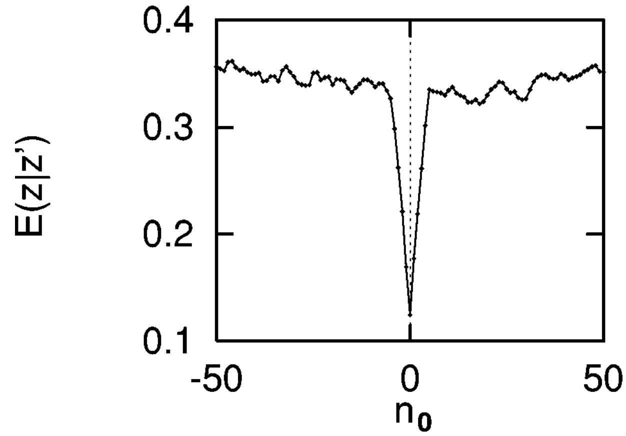

FIG. 4. The conditional entropy of phase synchronization in the Lorenz system where $\Delta t = 10^{-4}$.

where $u=5.7$, $\sigma=16$, $b=4$, and $r=45.92$. The response system is coupled to the drive system only through the scalar forcing term $x(t)$. g characterizes the strength of the unidirectional coupling. Now we discuss the conditional entropy of time series z and z'. Make the partition plane for the Rössler system $z_c=0.05$. Because the shape of the Lorenz attractor will change when generalized synchronization is implemented, we make a different partition plane for different g in the Lorenz system. Figure 1 shows the phase portrait of the Lorenz system after transients die away when $g=10$. Obviously, the two leaves of the Lorenz attractor become one. Figure 2 shows the results of conditional entropy after transients die out. From Fig. 2 one can see that there is a sharp minimum when $g=4.9$ and $g=10$, which correspond to generalized synchronization, and not when $g=0$ and $g=3$, which correspond to no generalized synchronization. Figures 2(a) and 2(b) just give some oscillation. The different oscillatory amplitudes between $n_0>0$ and $n_0<0$ come from the different oscillatory frequency between the Rössler and Lorenz systems. On the other hand, with the increasing of parameter g, the amplitudes of conditional entropy $E(z|z')$ decrease. So we can say that the appearance of the sharp minimum represents the existence of generalized synchronization. To confirm this, we have computed the largest conditional Lyapunov exponent of the response system, conditioned on the drive $x(t)$. Figure 3 shows the result. From Fig. 3 one can see that the cases of $g=4.9$ and 10 are located in the regime where the largest conditional Lyapunov exponents are negative, but the cases of $g=3$ and 0 are located in the regime where the largest conditional Lyapunov exponents are positive. For other coupling strength g, our numerical simulation gives similar results. So the case where there is a sharp minimum of conditional entropy corresponds to that where the largest conditional Lyapunov exponent of the response system is negative.

Second, we consider a simple example of phase synchronization, that is, two Lorenz systems with the parameters for one system in the chaotic regime $(r_1=28, b_1=8/3, \sigma_1=10)$ and for the other in the periodic regime $(r_2=270, b_2=8/3, \sigma_2=10)$. Following Ref. [5], at each time step the system $(i=1,2)$ is first integrated one time step via the flow equations. Then they interact with each other as follows:

$$0.94x_1+0.06x_2 \mapsto x_1, \quad 0.24x_1+0.76x_2 \mapsto x_2. \tag{7}$$

Letting time step $\Delta t=10^{-4}$ and taking two time series z_1 and z_2, one can get the conditional entropy $E(z_1|z_2)$. Figure 4 shows the result of the conditional entropy as a function of a shift parameter n_0. Obviously, it has a sharp minimum. So it is consistent with Ref. [5]. Comparing Fig. 4 with Fig. 2 one can see that the minimum in Fig. 4 is sharper than that in Fig. 2. That is because Fig. 4 represents the generalized synchronization of the same dynamics but Fig. 2 represents that of different dynamics. If we discuss the complete synchronization of Ref. [1], we will find that the sharp minimum becomes zero.

We have also studied the effect of external noise. We consider the Gaussian white noise ξ having zero mean and standard deviation equal to one, generated by using the Box-Müller method [11], and introduce noise in the form

$$x'=x'(1.0+\rho\xi), \tag{8}$$

where ρ denotes the intensity of external noise. This noise is applied at each integral step. Figure 5 shows the result corresponding to Fig. 2(d). From Fig. 5 one can see that the

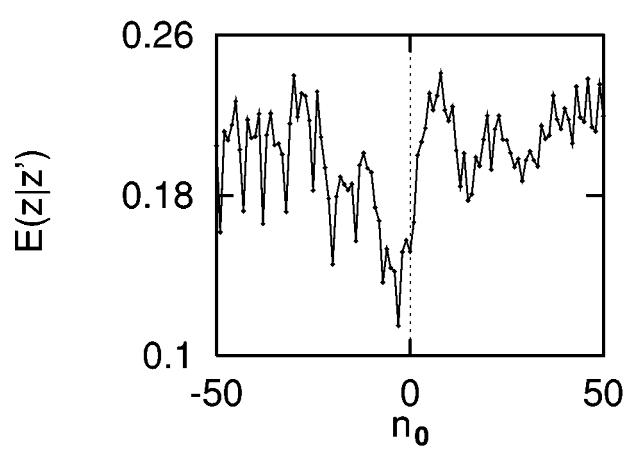

FIG. 5. The effect of noise corresponding to Fig. 2(d). The intensity of noise is 1.0×10^{-3}.

sharp minimum still exists. So this symbolic analysis method is useful in the case of weak noise.

In conclusion, we have demonstrated the possibility of illustrating generalized synchronization by symbolic analysis. Generalized synchronization can be implemented when the sharp minimum of the conditional entropy as a function of a shift parameter n_0 exists. It is a convenient method of testing whether there is generalized synchronization between different systems.

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